Figure 1 presents the variations of the velocity, shear, enthalpy, enthalpy gradient, and the location of dividing streamline with pressure gradient parameter β for the case of a hot wall with $f_w = -10$. We note that, in this case, the velocity has an overshoot. The results show that the maximum overshoot is near the dividing streamline. Figure 2 presents the variations of the same quantities as in Fig. 1 with β, for a cold wall with $f_w = -10$. In both cases, the location of the dividing streamline η_0 increases as β decreases. As has been proved by Kassoy,[6] to lowest order, η_0 is proportinal to $\beta^{-1/2}$. This has been confirmed by the present calculations. Figure 3 presents the velocity and temperature profiles for a cold wall, $s_w = -0.5$, with $f_w = -10$, at a stagnation point $\beta = 0.5$. The agreement with Libby's[7] analytic result for inner inviscid solutions is excellent.

Conclusions

The calculations performed using the bidirectional shooting method indicate that it is possible to integrate the boundary-layer equations under conditions of massive blowing. Unlike the conventional shooting method, which is unstable when the blowing rate increases, the proposed method avoids the unstable direction, instead integrates from the dividing streamline in both directions, i.e., toward the boundary-layer edge and toward the wall. The method proposed to solve the three-point boundary conditions is capable of solving complex boundary-layer problems involving mass and energy balance on the surface.

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Derivation of Aerodynamic Kernel Functions

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Recently there has been considerable interest in improving upon the potential flow aerodynamic model for determining the pressure distribution on lifting surfaces. In the present Note the method of Fourier Transforms is used to determine the Kernel Function which relates the pressure to the (prescribed) downwash within the framework of the shear flow model of Ref. 1. Such a model is intended to allow for the effects of an aerodynamic boundary layer which is, of course, neglected in the potential flow model. For simplicity we consider incompressible, steady flow. However the added difficulties associated with compressibility and nonsteadiness are largely computational rather than conceptual. We first illustrate the proposed method by rederiving known results from potential flow theory.

Fleeter[2] has recently used Fourier Transform methods to determine the Kernel Function for potential flow past a two-dimensional cascade of airfoils. Miles[3] has previously used such methods for the wind-tunnel correction problem. Several investigators, including Miles, Yates, and the two present authors have considered nonlifting "thickness" problems.

Potential Flow

Two dimensions

The pressure satisfies Laplace's equation (in x and z)

$$\nabla^2 p = 0 \tag{1}$$

as well as boundary conditions

$$\text{finiteness at infinity} \tag{2}$$

$$\partial p/\partial z = -\rho_\infty U_\infty \partial w/\partial x \quad \text{on wing} \tag{3}$$

$$p = 0 \quad \text{off wing} \tag{4}$$

w is the downwash (for steady flow, it is simply the slope times the freestream velocity) of the airfoil. Denote by an * the Fourier Transform with respect to streamwise coordinate x, e.g.,

$$p^* \equiv \int_{-\infty}^{\infty} p e^{-i\alpha x} dx \tag{5}$$

Transforming Eq. (1) and solving, using Eqs. (2) and (3), one obtains (on $z = 0$, the plane of the airfoil)

$$p^*/\rho_\infty U_\infty^2 = (i\alpha/|\alpha|)w^*/U_\infty \tag{6}$$

If one knew the downwash everywhere, then we would invert Eq. (6) as it stands to obtain an explicit solution for p, i.e.,

$$\frac{p}{\rho_\infty U_\infty^2} = \int_{-\infty}^{\infty} A(x-\xi)\frac{w(\xi)}{U_\infty} d\xi \tag{7}$$

where

$$p(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} p^* e^{i\alpha x} d\alpha$$
$$A(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\alpha}{|\alpha|} e^{i\alpha x} d\alpha = -\frac{1}{\pi x} \tag{8}$$

This is the solution to the so-called aerodynamic "thickness" problem. However our interest here is in the "lifting" problem where we do not know w off the wing but we do know that $p = 0$ there. Hence Eq. (6) is rewritten

$$w^*/U_\infty = (-i|\alpha|/\alpha)p^*/\rho_\infty U_\infty^2 \tag{9}$$

Inverting

$$\frac{w}{U_\infty} = \int_{\text{chord of wing}} K(x-\xi)\frac{p(\xi)}{\rho_\infty U_\infty^2} d\xi \tag{10}$$

where

$$K(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} -i\frac{|\alpha|}{\alpha} e^{i\alpha x} d\alpha = \frac{1}{\pi x} \tag{11}$$

(The similarity in form of A and K is fortuitous and does not usually extend to higher dimensions or the inclusion of compressibility or nonsteady effects.) Standard methods exist for solving integral equations such as Eq. (10). The present method has its usefulness in devising a means for determining the form of K.

Three dimensions

Defining a two-dimensional transform

$$p^* \equiv \iint_{-\infty}^{\infty} p(x, y, z)e^{-i\alpha x - i\gamma y} dx\, dy \tag{12}$$

we may solve the three-dimensional version of Eq. (1) subject to Eqs. (2–4). The results are

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$$\frac{w^*}{U_\infty} = \left\{-i\frac{[\alpha^2+\gamma^2]^{1/2}}{\alpha}\right\}\frac{p^*}{\rho_\infty U_\infty{}^2} \tag{13}$$

and, inverting

$$\frac{w(x,y)}{U_\infty} = \iint_{\text{Area of wing}} K(x-\xi, y-\eta)\frac{p(\xi,\eta)}{\rho_\infty U_\infty{}^2}d\xi\, d\eta \tag{14}$$

where

$$\begin{aligned} K(x,y) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\left\{-i\frac{[\alpha^2+\gamma^2]^{1/2}}{\alpha}\right\}e^{i\alpha x+i\gamma y}d\alpha\, d\gamma \\ &= \frac{1}{2\pi}\frac{1}{y^2}\left[1+\frac{x}{(x^2+y^2)^{1/2}}\right] \end{aligned} \tag{15}$$

Shear Flow in Two Dimensions

In the shear flow model, we allow the mean flow velocity $\bar{u}$ to be a function of the coordinate perpendicular to the plane of the airfoil z within a certain distance, the boundary-layer thickness δ. In Ref. 1 it is shown that the perturbation pressure p satisfies the field equation

$$\begin{aligned}(M_\infty{}^2/\bar{T}/T_\infty)D^3p - D\nabla^2 pU_\infty{}^2 + 2(\partial^2 p/\partial x\,\partial z)(d\bar{u}/dz)U_\infty{}^2 - \\ D(\partial p/\partial z)(1/\bar{T})(d\bar{T}/dz)U_\infty{}^2 = 0\end{aligned} \tag{16}$$

where

$$D \equiv \partial/\partial t + \bar{u}\,\partial/\partial x$$

with $\bar{u}(z)$ prescribed and $\to U_\infty$ for $z > \delta$. The mean temperature $\bar{T}$ is related to $\bar{u}$ through the energy equation for the mean flow.[1] In general, to determine K^* from Eq. (16) will require numerical procedures such as those described in Ref. 1. A subsequent numerical integration using the Fourier Transform inversion formula will determine K.

$$K = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} K^* e^{+i\alpha x+i\gamma y}d\alpha\, d\gamma \tag{17}$$

Here, for simplicity, we restrict ourselves to incompressible flow $M_\infty = 0$, $\bar{T} = T_\infty$, steady flow $\partial/\partial t = 0$, and two dimensions. Moreover we shall use an analytical asymptotic expression for K^* developed by Ventres[3] for small $\alpha\delta$. (Ventres has also included the effects of compressibility, nonsteadiness and three dimensionality.) For a power law velocity profile

$$\begin{aligned}\bar{u}/U_\infty &= (z/\delta)^{1/7} \quad z < \delta \\ &= 1 \quad z > \delta\end{aligned}$$

Ventres has determined that

$$K^* \cong -(i\alpha/|\alpha|)[1-\tfrac{28}{45}|\alpha|\delta] \tag{18a}$$

$$\cong -(i\alpha/|\alpha|)[1+\tfrac{28}{45}|\alpha|\delta] \tag{18b}$$

to consistent order in $\alpha\delta$.

Either of these expressions can be inverted analytically to provide an approximate Kernel Function for shear flows. The Fourier inverse of Eq. (18b) is

$$K(x) = (1/\pi x) - \tfrac{28}{45}\delta\mathscr{D}(x) \tag{19}$$

where $\mathscr{D}(x)$ is the doublet function. That is, for any continuous function $f(x)$

$$\int_{-\infty}^{\infty}\mathscr{D}(x-\xi)f(\xi)d\xi = df(x)/dx$$

Alternatively, the Fourier inverse of Eq. (18a) is

$$\begin{aligned} K(x) &= -\frac{\cos(x/\delta_n)}{\pi\delta_n}[\pi/2+Si(x/\delta_n)] + \frac{\sin(x/\delta_n)}{\pi\delta_n}Ci(x/\delta_n); \quad x>0 \\ &= \frac{\cos(x/\delta_n)}{\pi\delta_n}[\pi/2+Si(-x/\delta_n)] + \frac{\sin(x/\delta_n)}{\pi\delta_n}Ci(-x/\delta_n); \quad x<0 \end{aligned} \tag{20}$$

Here we have used $\delta_n = \frac{28}{45}\delta$. The sine and cosine functions Si and Ci are defined in Ref. 5.

The transformed Kernel's $K^*(\ )$ as given in Eqs. (18a) and (18b) can also be used to calculate the pressure on a wavy wall. If the wall deflection is given by

$$w = Re(w^* e^{i\alpha x})$$

then the corresponding pressure loading is $p = Re(p^* e^{i\alpha x})$ ($Re \equiv$ real part of). The complex amplitude p^* of the pressure is obtained from

$$\frac{p^*}{\rho_\infty U_\infty{}^2} = \frac{w^*/U_\infty}{K^*}$$

so that actually the reciprocal of Eq. (18a) or (18b) is used for this purpose. Yates[6] has developed expressions similar to Eqs. (18a) and (18b) for compressible flows, and has shown that the reciprocal of Eq. (18b) provides excellent agreement with experimental results[7] for high subsonic and low supersonic flows, when $\delta\alpha$ is as large as π. The boundary-layer thickness is then one half the wavelength of the sinusoidal wall deflection. On the other hand, Ventres[4] has shown that an expression equivalent to Eq. (18a) compares well with the same experimental data only for very small $\alpha\delta$. This indicates, but certainly does not conclusively prove, that Eq. (19) is the best form to use for $K(x)$.

An improvement over either Eqs. (18a) or (18b) could be obtained by solving Eq. (16) for large values of $\alpha\delta$, and constructing from that solution an approximate $K^*(\alpha)$ that would have the proper asymptotic behavior at both large and small $\alpha\delta$. The feasibility of this approach is being considered at the present time.

Asymmetry of Velocity Profiles on Upper and Lower Surfaces

In the preceding we have implicitly assumed that the mean flow over the upper and lower surfaces is the same. This is always true in the potential flow model but not necessarily for a shear flow. If this is not the case, then the above must be generalized. In the transform plane

$$\begin{aligned} w_U{}^*/U_\infty &= K_U{}^*(p_U{}^*/\rho_\infty U_\infty{}^2) \\ w_L{}^*/U_\infty &= -K_L{}^*(p_L{}^*/\rho_\infty U_\infty{}^2) \end{aligned} \tag{21}$$

where $K_U{}^* = K^*(\delta = \delta_U)$, $K_L{}^* = K^*(\delta = \delta_L)$; U, L – upper lower surface. For "lifting" surfaces $w_U{}^* = w_L{}^* = w^*$, hence from Eq. (21)

$$w^*/U_\infty = K_L{}^*K_U{}^*/(K_U{}^*+K_L{}^*)(p_U{}^*-p_L{}^*)/\rho_\infty U_\infty{}^2$$

or

$$\frac{w(x,y)}{U_\infty} = \iint \tilde{K}(x-\xi, y-\eta)\frac{[p_U-p_L]}{\rho_\infty U_\infty{}^2}(\xi,\eta)\,d\xi\, d\eta \tag{22}$$

where

$$\tilde{K}(x,y) \equiv \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{K_L{}^*K_U{}^*}{K_U{}^*+K_L{}^*}e^{i\alpha x+i\gamma y}d\alpha\, d\gamma$$

For the particular case where Eq. (18) applies, to lowest order in δ

$$\tilde{K} = \frac{K[\delta = (\delta_U+\delta_L)/2]}{2} \tag{23}$$

a physically appealing result.

Further Studies

Using the asymptotic theory of Ventres and Yates, other Kernel Functions for compressible, nonsteady, three-dimensional flows may be derived. In such cases K^* is known analytically and frequently K may also be so determined. For some, perhaps most, applications such K^* and K will not be quantitatively accurate. Then one must resort to numerical procedures[1] to determine K^* and evaluate K by numerical integration of the Fourier Inversion formula.

In some applications it will also be of interest to allow for the (slowly) varying mean flow with streamwise coordinate x. This requires a conceptual advance; however, one should be able to take advantage of the slowly varying nature of the flow velocity field with x.

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Unsteady Flow of Power-Law Fluids

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Introduction

SOLUTIONS for power-law fluids have been presented in Refs. 1 and 2 for both impulsively started plate and flow cases. In Ref. 1 a two-point boundary value problem was formulated and solved while in Ref. 2 a perturbation technique was used, the accuracy of which is limited to fluids which are slightly non-Newtonian. In the present Note we present a noniterative solution to the same problem for all power-law fluids formulated as an initial value rather than a boundary value problem. The formulation results in a simple expression which readily can be integrated in closed form with such forms available in integration tables.

Analysis

The differential equation pertaining to the problem has been derived before[1] and it is in the form

$$G_{\eta\eta} = -2\eta(G_\eta)^{2-N} \tag{1}$$

subject to the boundary conditions

$$G(0) = 0, \qquad G(\infty) = 1 \tag{2}$$

Let us introduce the following transformation[3] for Eqs. (1) and (2):

$$\eta = \bar{\eta}A^{\alpha_1}, \qquad G = \bar{G}A^{\alpha_2} \tag{3}$$

We obtain then the following equivalent problem:

$$\bar{G}_{\bar{\eta}\bar{\eta}} = -2\bar{\eta}(\bar{G}_{\bar{\eta}})^{2-N} \tag{4}$$

subject to the initial conditions

$$\bar{G}(0) = 0, \qquad \bar{G}_{\bar{\eta}}(0) = 1 \tag{5}$$

with

$$A = [\bar{G}(\infty)]^{-2/(N+1)}, \quad \eta = A^{(N-1)/2}\bar{\eta}, \quad \bar{G} = A^{(N+1)/2}\bar{G} \tag{6}$$

As $N = 1$ represents the Newtonian case, we analyse cases with $N \neq 1$. Integrating Eq. (4) once using the second condition in Eq. (5) yields

$$(\bar{G}_{\bar{\eta}})^{N-1} = -(N-1)\bar{\eta}^2 + 1$$

or

$$(\bar{G}_{\bar{\eta}}) = [1-(N-1)\bar{\eta}^2]^{1/(N-1)} \tag{7}$$

Integrating Eq. (7) using the first condition in Eq. (5) results in the following expression for $\bar{G}(\bar{\eta})$:

$$\bar{G}(\bar{\eta}) = \int_0^{\bar{\eta}} [1-(N-1)\bar{\eta}^2]^{1/(N-1)}\, d\bar{\eta} \tag{8}$$

Closed form solutions to Eq. (8) are available for a wide range of N such as $N = \frac{1}{3}; \frac{1}{2}; \frac{2}{3}; \frac{3}{4}$; 1.25; 1.5; 2; 3.

As in the work of Ref. 1 we define $\bar{\eta}_\infty$ for $N > 1$ to be the value of $\bar{\eta}$ which makes the integrand in Eq. (8) zero, i.e.,

$$1-(N-1)\bar{\eta}_\infty{}^2 = 0$$

which yields

$$\bar{\eta}_\infty = [1/(N-1)]^{1/2} \quad \text{for} \quad N > 1 \tag{9}$$

For $N < 1$, $\bar{\eta}_\infty$ is taken to be the value of $\bar{\eta}$ where $\bar{G}(\infty)$ reaches a constant asymptotic value.

Equation (8) was integrated from $\bar{\eta} = 0$ to various values of $\bar{\eta}$ including $\bar{\eta}_\infty$. The value of $\bar{G}(\infty)$ thus obtained was used to obtain the value of A and the relations between η and $\bar{\eta}$ and G and $\bar{G}$ which are given in Eq. (6). For those values of N to which closed form solutions are available, the results were obtained using such solutions. Solutions for other values of N were obtained by the numerical integration of Eq. (8).

Results and Conclusions

In Table 1 we compare the results obtained using the present method with the results presented in Refs. 1 and 2. The results compare the drag coefficient $C_f(N)$ expressed as

$$C_f(N) = [G_\eta(0)]^N/[2N(N+1)]^{N/(N+1)} \tag{10}$$

The table shows that the present method yields results with high degree of accuracy for all values of N. The results from the perturbation solution is reliable for values of N in the vicinity of 1. Furthermore, we believe that the present initial value method yields much simpler expressions in the analysis than those presented in Ref. 1 which were based upon an analysis of a boundary value problem.

Table 1 Numerical values of skin-friction coefficient $C_f(N)$

N	Present method	Ref. 1	Roy[2]	Wells[4]	Ref. 5, source of closed form solution
0.25	0.9892	0.9925	1.0099		
0.50	0.8128	0.8145	0.8219	0.816	p. 16
0.75	0.6711	0.6718	0.6727		p. 36
1.00	0.5642	0.564	0.5642	0.564	
1.25	0.4807	0.4823	0.4815		Direct analytical integration
1.50	0.4171	0.4187	0.4123		Direct analytical integration
1.75	0.3677	0.3683	0.3483		
2.0	0.3269	0.3276	0.2843		Direct analytical integration

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